On the principles of B-smooth discontinuous flows

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Abstract

In this paper, we define $B-$ smooth discontinuous dynamical systems which can be used as models of various processes in mechanics, electronics, biology and medicine. We find sufficient conditions to guarantee the existence of such systems. These conditions are easy to verify. Appropriate examples are constructed.

1 Introduction and Preliminaries

A book [1] edited by D.V. Anosov and V.I. Arnold considers two fundamentally different Dynamical Systems (DSs) : flows and cascades. Roughly speaking, flows are DSs with continuous time and cascades are DSs with discrete time. One of the most important theoretical problem is to consider *Discontinuous Dynamical Systems* $(DDSs)$. That is systems whose trajectories are piecewise

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continuous curves. It is well-recognized (for example, see [2]) that the general notion of such systems was introduced by Th. Pavlidis [3]-[5], although particular examples (the mathematical model of clock [6]-[8] and so on) had been discussed before. Some basic elements of the theory are given in [9]-[12]. Analysing the behavior of the trajectories we can conclude that $DDSs$ combine features of vector fields and maps, they can not be reduced to flows or cascades, but are close to flows since time is continuous. That is why we propose to call them also *Discontinuous Flows* (DFs). Applications of $DDSs$ in mechanics, electronics, biology and medicine were considered in [3]-[5], [13] - [16]. Chaotic behavior of discontinuous processes was investigated in [14, 17]. One must emphasize that DFs are not *differential equations with discontinuous right side* which often have been accepted as $DDSs$ [18]. However, theoretical problems of nonsmooth dynamics and discontinuous maps [19]-[27] are also very close to the subject of our paper. One should also agree that *nonautonomous impulsive differential equations*, which were thoroughly described in [10] and [12], are not DFs.

Papers of T. Pavilidis and V. Rozhko [3]-[5], [11] contain interesting practical and theoretical ideas concerning the DFs. They formulated some important conditions on differential equations, but not all of them were used to prove basic properties of DFs. Some aspects of DFs on manifolds were considered in [28]. One must remark that the authors of the paper formulated conditions for the group property, but as is demonstrated by example 9.3 of our paper those conditions do not guarantee it. In that paper the *smooth impulsive flow* was claimed to be considered, but differentiable dependence, as well as continuous dependence, were not defined and investigated. Thus one can say that the complexity of DFs necessitates more careful investigation. And our article can be considered as an attempt to give a rigorous description of DFs.

The paper embodies results that provide sufficient conditions for the exis-

tence of a *differentiable* DF. Since DFs have specific smoothness of solutions we call these systems *B-differentiable DFs.* Apparently, it is the first time when notions of $B-$ continuous and $B-$ differentiable dependence of solutions on initial values [30, 31] are applied to described $DDSs$ and sufficient conditions for the continuation of solutions and the group property are obtained. A central auxiliary result of the paper is the construction of a new form of the general autonomous impulsive equation (system (1)). Effective methods of investigation of systems with variable time of impulsive actions were considered in [10, 12], [30]- [35].

Let $\mathbb{Z}, \mathbb{N}$ and $\mathbb{R}$ be the sets of all integers, natural and real numbers, respectively. Denote by $||\cdot||$ the Euclidean norm in $\mathbb{R}^n$, $n \in \mathbb{N}$. Consider a set of strictly ordered real numbers $\{\theta_i\}$, where the set $\mathcal{A}$ of indices is an interval of $\mathbb{Z}/\{0\}$.

DEFINITION 1.1 *The set* $\{\theta_i\}$ *is said to be a sequence of* $\beta - type$ *if the product* $\theta_i, \quad i \geq 0$ *for all* i *and one of the following alternative cases holds:*

(a) $\{\theta_i\} = \emptyset$*;*

(b) $\{\theta_i\}$ *is a finite and nonempty set;*

(c) $\{\theta_i\}$ *is an infinite set such that* $|\theta_i| \to \infty$ *as* $|i| \to \infty$.

From the definition, it follows immediately that a sequence of $\beta - type$ does not have a finite accumulation point in $\mathbb{R}$.

DEFINITION 1.2 *A function* $\varphi : \mathbb{R} \longrightarrow \mathbb{R}^n$ *is said to be from a space* $\mathcal{PC}(\mathbb{R})$ *if*

1. $\varphi(t)$ *is left continuous on* $\mathbb{R}$;
2. *there exists a sequence* $\{\theta_i\}$ *of* $\beta-$ *type such that* φ *is continuous if* $t \neq \theta_i$ *and* φ *has discontinuities of the first kind at the points* θ_i.

Particularly, $C(\mathbb{R}) \subset \mathcal{PC}(\mathbb{R})$.

DEFINITION 1.3 *A function $\varphi(t)$ is said to be from a space $\mathcal{PC}^1(\mathbb{R})$ if $\varphi' \in \mathcal{PC}(\mathbb{R})$.*

Let T be an interval in $\mathbb{R}$.

DEFINITION 1.4 *We denote by $\mathcal{PC}(T)$ and $\mathcal{PC}^1(T)$ the sets of restrictions of all functions from $\mathcal{PC}(\mathbb{R})$ and $\mathcal{PC}^1(\mathbb{R})$ on T respectively.*

Let G be an open subset of $\mathbb{R}^n$, G_r be an $r-$ neighbourhood of G in $\mathbb{R}^n$ for a fixed $r > 0$ and $\hat{G} \subset G_r$ be an open subset of $\mathbb{R}^n$. Denote as $\Phi : \hat{G} \longrightarrow \mathbb{R}$ be a function from $C^1(\hat{G})$ and assume that a surface $\Gamma = \Phi^{-1}(0)$ is a subset of $\bar{G}$, where $\bar{G}$ denotes the closure of the set G in $\mathbb{R}^n$. Moreover, define a function $J : \Gamma_r \to \bar{G}$, where Γ_r is an $r-$ neighbourhood of Γ. We shall need the following assumptions.

C1) $\nabla\Phi(x) \neq 0$, $\forall x \in \Gamma$;

C2) $J \in C^1(\Gamma_r), \det[\frac{\partial J(x)}{\partial x}] \neq 0$, for all $x \in \Gamma$.

One can see that the restriction $J|_\Gamma$ is a one-to-one function. Let also $\tilde{\Gamma} = J(\Gamma)$, $\tilde{\Gamma} \subset \bar{G}$. If $\tilde{\Phi}(x) = \Phi(J^{-1}(x))$, $x \in \tilde{\Gamma}$ then $\tilde{\Gamma} = \left\{x \in G |\, \tilde{\Phi}\,(x) = 0\right\}$. It is easy to verify that $\nabla\tilde{\Phi}(x) \neq 0, \forall x \in \tilde{\Gamma}$. Condition $C1)$ implies that for every $x_0 \in \Gamma$ there exists a number $j = \overline{1, n}$ and a function $\varphi_{x_0}(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$ such that in a neigbourhood of x_0 the surface Γ is the graph of the function $x_j = \varphi_{x_0}(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$. The same is true for every $x_0 \in \tilde{\Gamma}$.

Remark 1.1 One can see from the description of Γ and $\tilde{\Gamma}$ that the surfaces are C^1 boundaryless $n-1$ dimensional manifiods [37].

Consider the following impulsive differential equation in the domain $D = G \cup \Gamma \cup \tilde{\Gamma}$.

$$x'(t) = f(x(t)), \{x(t) \notin \Gamma \wedge t \geq 0\} \vee \{x(t) \notin \tilde{\Gamma} \wedge t \leq 0\},$$

$$
\begin{aligned}
&x(t+)|_{x(t-)\in\Gamma\wedge t\geq 0} = J(x(t-)),\\
&x(t-)|_{x(t+)\in\tilde{\Gamma}\wedge t\leq 0} = J^{-1}(x(t+)).
\end{aligned}
\tag{1}
$$

Denote by ∂A the boundary of a set A. We make the following assumptions which will be needed throughout the paper.

C3) $f \in C^1(G_r)$.

C4) $\Gamma \cap \tilde{\Gamma} = \emptyset$.

C5) $\langle \nabla\Phi(x), f(x)\rangle \neq 0$ if $x \in \Gamma \cup \partial\Gamma$ and $f(x) \neq 0$.

C6) $\left\langle \nabla\tilde{\Phi}(x), f(x)\right\rangle \neq 0$ if $x \in \tilde{\Gamma} \cup \partial\tilde{\Gamma}$ and $f(x) \neq 0$.

2 Existence and Uniqueness

Definition 2.1 *A function $x(t) \in \mathcal{PC}^1(T)$ with a set of discontinuity points $\{\theta_i\} \subset T$ is said to be a solution of (1) on the interval $T \subset \mathbb{R}$ if it satisfies the following conditions:*

(i) equation (1) is satisfied at each point $t \in T\backslash\{\theta_i\}$ and $x'(\theta_i-) = f(x((\theta_i))), i \in \mathcal{A}$, where $x'(\theta_i-)$ is the left-sided derivative;

(ii) $x(\theta_i+) = J(x((\theta_i))$ for all θ_i.

Theorem 2.1 *Assume that conditions $C1) - C6)$ hold. Then for every $x_0 \in D$ there exists an interval $(a, b) \subset \mathbb{R}, a < 0 < b$, such that the solution $x(t) = x(t, 0, x_0)$ of (1) exists on the interval.*

Proof. To prove the theorem we consider the following several cases.

(a) Assume that $x_0 \notin \Gamma \cup \tilde{\Gamma} \cup\ \partial\Gamma \cup \partial\tilde{\Gamma}$. Then there exists a number $\epsilon > 0$ such that $B(x_0, \epsilon) \cap (\Gamma \cup \tilde{\Gamma}) = \emptyset$. Therefore, by the existence and uniqueness

theorem [36], the solution exists and is unique on an interval (a, b) as a solution of the system

$$y' = f(y). \tag{2}$$

(b) If $x_0 \in \Gamma$ then $x(0+) \in \tilde{\Gamma}$. Assume that $x(0+) \in \partial\Gamma$, then by $C5$) the solution can be continued continuously to a moment $b > 0$. If $x(0+) \notin \partial\Gamma$, then there exists a number $\epsilon > 0$ such that $B(x(0+), \epsilon) \cap \Gamma = \emptyset$ and x(t) can be continued again to the right continuously. Let us consider decreasing t now. If $x_0 \in \partial\tilde{\Gamma}$ then by $C6$) x(t) exists on an interval $(a, 0]$ and is continuous .

If $x_0 \notin \partial\tilde{\Gamma}$ then there exists $\epsilon > 0$ such that $B(x_0, \epsilon) \cap \tilde{\Gamma} = \emptyset$ and, hence, $x(t)$ can be continued to a moment $a < 0$ as a solution of (2).

(c) We can discuss the case $x_0 \in \tilde{\Gamma}$ similarly to the previous one.

(d) Assume that $x_0 \in \partial\Gamma \backslash \partial\tilde{\Gamma}$. If $x_0 \in \tilde{\Gamma}$ then (c) implies that $x(t)$ is continuable to a moment $a < 0$. Othewise there exists a number $\epsilon > 0$ such that $B(x_0, \epsilon) \cap \tilde{\Gamma} = \emptyset$. Hence, the solution $x(t)$ can be continued to a moment $a < 0$ as a solution of (2) without a meeting with $\tilde{\Gamma}$. Condition C5) implies that $x(t)$ can be continued to a moment $b > 0$ as a solution of (2) again without any meeting with Γ.

(c) The case $x_0 \in \partial\tilde{\Gamma} \backslash \partial\Gamma$ can be considered similarly to the previous one.

(e) Assume that $x_0 \in \partial\tilde{\Gamma} \cap \partial\Gamma$. Conditions $C5$) and $C6$) imply that there exist real numbers $a, b, a < 0 < b$, such that x(t) is a continuous solution of (1) on the interval (a, b) and is unique.

It is obvious that the existence is valid if $f(x_0) = 0$. The uniqueness of the solution for all cases $(a)-(f)$ follows from the theorem on uniqueness for ordinary differential equations [36] and invertibility of the function J.

3 Continuation of solutions

Definition 3.1 *A solution* $x(t) : [a, \infty) \to \mathbb{R}^n, a \in \mathbb{R}$, *of (1) is said to be continuable to* ∞.

Definition 3.2 *A solution* $x(t) : (-\infty, b] \to \mathbb{R}^n, b \in \mathbb{R}$, *of (1) is said to be continuable to* $-\infty$.

Definition 3.3 *A solution* $x(t)$ *of (1) is said to be continuable on* $\mathbb{R}$ *if it is continuable to* ∞ *and to* $-\infty$.

Definition 3.4 *A solution* $x(t) = x(t, 0, x_0)$ *of (1) is said to be continuable to a set* $S \subset \mathbb{R}^n$ *as time decreases (increases) if there exists a moment* $\xi \in \mathbb{R}$, *such that* $\xi \leq 0\,(\xi \geq 0)$ *and* $x(\xi) \in S$.

Denote by $B(x_0, \xi) = \{x \in \mathbb{R}^n | ||x - x_0|| < \xi\}$ a ball with centre $x_0 \in \mathbb{R}^n$ and radius $\xi \in \mathbb{R}$.

The following Theorem provides sufficient conditions for the continuation of solutions of (1).

Theorem 3.1 *Assume that*

(a) Every solution $y(t, 0, x_0), x_0 \in D$, *of (2) is either continuable to* ∞ *or continuable to* Γ *as time increases;*

(b) For every $x \in \tilde{\Gamma}$ *there exists a number* ϵ_x *such that* $\bar{B}(x, \epsilon_x) \cap \Gamma = \emptyset$

(c) $\inf_{(x,\epsilon_x)\in\Gamma\times(0,\infty)} \frac{\epsilon_x}{\sup_{B(x,\epsilon_x)} ||f(x)||} = \theta > 0$

Then every solution $x(t) = x(t, 0, x_0), x_0 \in D$, *of (1) is continuable to* ∞.

Proof. Let $x(\theta_i+) \in \tilde{\Gamma}$ for fixed i. Assume that there exists a number $\xi > \theta_i$ such that $||x(\xi) - x(\theta_i+)|| = \epsilon_{x(\theta_i+)}$ (otherwise $x(t)$ is continuable to ∞). Then

$$x(\xi) = x(\theta_i+) + \int_{\theta_i}^{\xi} f(x(s))ds,$$

and $\epsilon_{x(\theta_i+)} \leq M_{x(\theta_i+)}\left(\xi-\theta_i\right) \leq M_{x(\theta_i+)}\left(\theta_{i+1}-\theta_i\right)$. The last inequality implies that $\theta_{i+1}-\theta_i \geq \theta$ for all i. That is θ_i is a sequence of $\beta-$ type if $\theta_i \geq 0$.

In a similar manner one can prove that the following theorem is valid.

THEOREM 3.2 *Assume that:*

(a) every solution $y(t,0,x_0)$, $x_0 \in D$, of (2) is continuable either to $-\infty$ or to $\tilde{\Gamma}$ as time decreases;

(b) for every $x \in \Gamma$ there exists a number $\epsilon_x > 0$ such that $\bar{B}(x,\epsilon_x) \cap \tilde{\Gamma} = \emptyset$;

(c) $\inf_{(x,\epsilon_x)\in\tilde{\Gamma}\times(0,\infty)} \frac{\epsilon_x}{\sup_{B(x,\epsilon_x)}\|f(x)\|} = \theta > 0.$

Then every solution $x(t) = x(t,0,x_0), x_0 \in D$, of (1) is continuable to $-\infty$.

Theorems 3.1 and 3.2 imply that the following assertion is valid.

THEOREM 3.3 *Assume that*

(a) every solution $y(t,0,x_0), x_0 \in D$, of (2) satisfies the following conditions:

(a1) it is continuable either to ∞ or to Γ as time increases,

(a2) it is continuable either to $-\infty$ or to $\tilde{\Gamma}$ as time decreases;

(b) for every $x \in \tilde{\Gamma}$ there exists a number $\epsilon_x > 0$ such that $\bar{B}(x,\epsilon_x) \cap \Gamma = \emptyset$;

b′) for every $x \in \Gamma$ there exists a number $\tilde{\epsilon}_x > 0$ such that $\bar{B}(x,\tilde{\epsilon}_x) \cap \tilde{\Gamma} = \emptyset$;

(c) $\inf_{(x,\epsilon_x)\in\tilde{\Gamma}\times\mathbb{R}} \frac{\epsilon_x}{\sup_{\bar{B}(x,\epsilon_x)}\|f(x)\|} > 0;$

c′) $\inf_{(x,\tilde{\epsilon}_x)\in\Gamma\times R} \frac{\tilde{\epsilon}_x}{\sup_{B(x,\tilde{\epsilon}_x)}\|f(x)\|} > 0.$

Then every solution $x(t) = x(t,0,x_0)$, $x_0 \in D$, of (1) is continuable on $\mathbb{R}$.

Let us introduce a distance between two sets $A, B \subset \mathbb{R}^n$ as $dist(A,B) = \inf\{||a-b|| \,|\, a \in A, b \in B\}$.

Other sufficient conditions for the continuation of solutions of (1) are provided by the following theorems.

Theorem 3.4 *Assume that*

(a) Every solution $y(t,0,x_0)$, $x_0 \in D$, *of (2) satisfies the following conditions:*

(a1) it is continuable either to ∞ *or to* Γ *as* t *increases;*

(a2) it is continuable either to $-\infty$ *or to* $\tilde{\Gamma}$ *as* t *decreases;*

(b) $\sup_D |f(x)| < +\infty$*;*

(c) $dist(\Gamma, \tilde{\Gamma}) > 0$.

Then a solution $x(t,0,x_0), x_0 \in D$, *of (1) is continuable on* $\mathbb{R}$.

Proof. Fix $x_0 \in D$ and let $x(t) = x(t,0,x_0)$ be the solution of (1). According to Definition 1.1 we shall consider the following three cases:

A) If x(t) is a continuous solution of (1), then it is a solution of (2) and, hence is continuable on $\mathbb{R}$.

B) Denote by $\theta_{\max}$ and $\theta_{\min}$ the maximal an minimal elements of the set $\{\theta_i\}$ respectively. Consider $t \geq \theta_{\max}$. By the condition on J the value $x(\theta_{\max}+) = J(x(\theta_{\max}-)) \in D$ and the solution $x(t) = y\,(t, \theta_{\max}, x(\theta_{\max}+))$, where y is the solution of (2) and is continuable to ∞. For $t \leq \theta_{\min}$ one can apply the same arguments to show that x(t) is continuable to $-\infty$.

C) Three alternatives exists. Let us consider them in turn.

c_1) If the sequence $\{\theta_i\}$ has a maximal element $\theta_{\max} \in R$, then using B) it is easy to prove that $x(t)$ is continuable to ∞. Let t be decreasing. We have that

$$x(\theta_i+) = x(\theta_{i+1}) + \int_{\theta_{i+1}}^{\theta_i} f(x(s))ds. \qquad (3)$$

Denote $\sup_D |f(x)| = M$ and $dist(\Gamma, \tilde{\Gamma}) = \alpha$. Then (3) impies that $\frac{\alpha}{M} \leq (\theta_{i+1} - \theta_i)$. Hence $\frac{\alpha}{M}(i - i_0) \geq (\theta_i - \theta_{i_0})$, where i_0 is fixed. The last inequality shows that $\theta_i \to -\infty$ as $i \to -\infty$. Thus, $x(t)$ is continuable to $-\infty$.

c_2) Assume that the sequence $\{\theta_i\}$ has a minimal element $\theta_{\min} \in R$. Then the arguments of B) indicate that $x(t)$ is continuable to $-\infty$. For increasing t we have that

$$x(\theta_{i+1}) = x(\theta_i+) + \int_{\theta_i}^{\theta_{i+1}} f(x(s))ds, \tag{4}$$

$\frac{\alpha}{M} \leq (\theta_{i+1} - \theta_i)$ or $\frac{\alpha}{M}(i - i_0) \leq (\theta_i - \theta_{i_0})$, where i_0 is fixed. Hence, $\theta_i \to \infty$ as $i \to \infty$. That is, $x(t)$ is continuable to ∞.

c_3) Assume that $\{\theta_i\}$ has neither a minimal nor a maximal element. The result for this case follows from c_1) and c_2). The proof is complete.

THEOREM 3.5 *Assume that*

(a) Every solution $y(t, 0, x_0), x_0 \in D,$ *of (2) is continuable either to* ∞ *or to* Γ *as time increases;*

(b) there exists a neighbourhood S *of* Γ *in* D *such that*

(b1) $dist(\Gamma, \partial S) > 0$;

(b2) $\sup_S \|f(x)\| < \infty$;

(b3) $\tilde{\Gamma} \cap S = \emptyset$.

Then every solution $x(t) = x(t, 0, x_0), x_0 \in D,$ *of (1) is continuable to* ∞.

Proof. Denote $d = dist(\Gamma, \partial S)$ and $M = \sup_S \|f(x)\|$. For fixed i one can see that

$$x(\theta_{i+1}) = x(\theta_i+) + \int_{\theta_i}^{\theta_{i+1}} f(x(s))ds.$$

Condition $b3$) implies that $d < |x(\theta_{i+1}) - x(\theta_i+)| \leq M(\theta_{i+1} - \theta_i)$. Thus $\theta_{i+1} - \theta_i \geq \frac{d}{M} > 0$ for all i. Further discussion is fully analogous to the proof of the last Theorem.

Similarly, one can prove that the following assertion is valid.

Theorem 3.6 *Assume that:*

(a) every solution $y(t, 0, x_0), x_0 \in D,$ *of (2) is continuable either to* $-\infty$ *or to* $\tilde{\Gamma}$ *as time decreases,*

(b) there exists a neighbourhood $\tilde{S}$ *of* $\tilde{\Gamma}$ *in* D *such that:*

(b1) $dist(\tilde{\Gamma}, \partial\tilde{S}) > 0$

(b2) $\sup_{\tilde{S}} \|f(x)\| < \infty$

(b3) $\Gamma \cap \tilde{S} = \emptyset$.

Then every solution $x(t) = x(t, 0, x_0), x_0 \in D,$ *of (1) is continuable to* $-\infty$.

Using the conditions of both Theorems 3.5 and 3.6 one can formulate the following assertion.

Theorem 3.7 *Assume that:*

(a) every solution $y(t, 0, x_0), x_0 \in D,$ *of (2) satisfies the following conditions:*

(a1) it is continuable either to ∞ *or to* Γ *as time increases;*

(a2) it is continuable either to $-\infty$ *or to* $\tilde{\Gamma}$ *as time decreases;*

(b) there exists a neighbourhoods S *and* $\tilde{S}$ *of* Γ *and* $\tilde{\Gamma}$ *in* D*, respectively, such that:*

(b1) $dist(\Gamma, \partial S) > 0$, $dist(\tilde{\Gamma}, \partial\tilde{S}) > 0$;

(b2) $\sup_{S\cup\tilde{S}} \|f(x)\| < \infty$;

(b3) $\tilde{\Gamma} \cap S = \emptyset$, $\Gamma \cap \tilde{S} = \emptyset$.

Then every solution $x(t) = x(t, 0, x_0), x_0 \in D,$ *of (1) is continuable on* $\mathbb{R}$.

4 The Group Property

Consider a solution $x(t) : \mathbb{R} \to \mathbb{R}^n$ of (1). Let $\{\theta_i\}$ be the sequence of discontinuity points of $x(t)$. Fix $\theta \in \mathbb{R}$ and introduce a function $\psi(t) = x(t+\theta)$.

Lemma 4.1 *The set $\{\theta_i - \theta\}$ is a set of all solutions of the equation*

$$\Phi(\psi(t)) = 0. \tag{5}$$

Proof. We have that $\Phi(\psi((\theta_i - \theta)) = \Phi(x((\theta_i - \theta) + \theta)) = \Phi(x(\theta_i)) = 0$. Assume that $t = \varphi$ is a solution of (5), then $\Phi(x(\varphi + \theta)) = \Phi(\psi(\varphi)) = 0$. That is, $\varphi + \theta$ is one of the numbers $\{\theta_i\}$. Let $\varphi + \theta = \theta_j$, then $\varphi = \theta_j - \theta$. The lemma is proved. The following condition is one of the main assumptions for DFs.

$C7)$ $\Gamma, \tilde{\Gamma} \subset \partial G$.

Lemma 4.2 *Assume that $C1) - C7)$ hold. Then $x(-t, 0, x(t, 0, x_0)) = x_0$ for all $x_0 \in D, t \in \mathbb{R}$.*

Proof. Consider only $t > 0$, as $t < 0$ is very similar to the first case and $t = 0$ is primitive. If the set $\{\theta_i\}$ is empty then proof follows immediately from the assertion for DS [1]. One can see that it remains to check the equality $x(\theta_i-, 0, x(\theta_i+)) = x(\theta_i)$ is valid for all i, and the condition $x(-\theta_1, 0, x(\theta_1, 0, x_0)) = x_0$ is fulfilled. The first one is obvious because of invertibility of J. Let us consider the second one. Denote $x(t) = x(t, 0, x_0), \tilde{x}(t) = x(t, 0, x(\theta_1))$. Since $x(\theta_1) \in \Gamma, \Gamma \cap \tilde{\Gamma} = \emptyset$, the solution $\tilde{x}$ moves along the trajectory of (2) for decreasing t. And it could not meet $\tilde{\Gamma}$ if $t > -\theta_1$. Indeed, assume on the contrary that there exists $\theta, -\theta_1 < \theta < 0$, moment where $\tilde{x}$ intersects $\tilde{\Gamma}$. Then $\tilde{x}(\theta+) = x(\theta + \theta_1) = x(\theta + \theta_1 +)$. As $x(\theta + \theta_1)$ is a point of $\tilde{\Gamma} \subset \partial D$ and $\tilde{\Gamma}$ is smooth,

then $x'(\theta+\theta_1)$ is tangent to $\tilde{\Gamma}$ at $\tilde{x}(\theta+)$. That is $\left\langle \nabla\tilde{\Phi}(\tilde{x}(\theta+)), f(\tilde{x}(\theta+))\right\rangle = 0$. Thus we have obtained contradiction with $C6$). The Lemma is proved.

LEMMA 4.3 *If $x(t): T \to \mathbb{R}^n$ is a solution of (1) then $x(t+\theta), \theta \in \mathbb{R}$, is also a solution of (1).*

Proof.

(a) From the last lemma it follows that $\psi = x(t+\theta)$ is continuous on the interval $(\theta_i - \theta, \theta_{i+1} - \theta], i \in \mathbb{Z}$. Fix $i \in \mathbb{Z}$, and consider $t \in (\theta_i - \theta, \theta_{i+1} - \theta]$. We have that $t + \theta \in (\theta_i, \theta_{i+1}]$ and in the same manner as for DSs one can verify that $\psi'(t) = f(\psi(t))$. That is, the equation (1) is satisfied by $x(t+\theta)$ for all $t \neq \theta_i - \theta$, $i \in \mathbb{Z}$, if we mean the left sided derivatives.

(b) For fixed i we have that $\psi(\theta_i - \theta+) = x(\psi(\theta_i - \theta+) + \theta) = x(\theta_i+) = J(x(\theta_i)) = J(\psi(\theta_i - \theta))$. Thus, one can see that the impulsive equation in (1) is also satisfied by $x(t+\theta)$ and this completes the proof.

Lemmas 4.1-4.3 imply that the following theorem is valid.

THEOREM 4.1 *Assume that conditions $C1) - C7)$ are fulfilled. Then*

$$x(t_2, x(t_1, x_0)) = x(t_2 + t_1, x_0), \tag{6}$$

for all $t_1, t_2 \in \mathbb{R}$.

5 Continuous dependence of solutions on initial values

Let $x^0(t): [a,b] \to \mathbb{R}^n, a \leq 0 \leq b$, be a solution of (1), $x^0(t) = x(t, 0, x_0), \theta_i, i = -k, \ldots, -1, 1, \ldots, m$, are the points of discontinuity of $x^0(t)$, such that $a \leq \theta_{-k} < \ldots < \theta_{-1} \leq 0 \leq \theta_1 < \ldots < \theta_m \leq b$. Denote by $x(t) = x(t, 0, \bar{x})$ another solution of (1).

DEFINITION 5.1 *The solution $x(t) : [a,b] \to \mathbb{R}^n$ is said to be in an ϵ-neighbourhood of $x^0(t)$ if:*

1. *every point of discontinuity of $x(t)$ lies in an ϵ-neighbourhood of a point of discontinuity of $x^0(t)$;*

2. *For each $t \in [a,b]$ which is outside of ϵ-neighbourhood of points of discontinuity of $x^0(t)$ the inequality $\|x^0(t) - x(t)\| < \epsilon$ holds.*

DEFINITION 5.2 *Hausdorff's topology, which is built on the basis of all ϵ-neighbourhoods, $0 < \epsilon < \infty$, of piecewise solutions will be called $B_{[a,b]}$-topology.*

THEOREM 5.1 *Assume that conditions $C1)-C7)$ are satisfied. Then the solution $x(t)$ continuously depends on initial value in $B_{[a,b]}$ topology .*
Moreover, if all $\theta_i, i = -k, \ldots, -1, 1, \ldots, m,$ are interior points of $[a,b]$, then, for sufficiently small $||x_0 - \bar{x}||$, the solution $x(t) = x(t, 0, \bar{x}), x(t) : [a,b] \to \mathbb{R}^n$, meets the surface Γ exactly $m + k - 1$ times.

Proof. We consider only the section $[0,b]$. The closeness of x(t) and $x^0(t)$ on $[a,0]$ can be considered similarly. There are two cases: a)$x_0 \in \Gamma; b)x_0 \notin \Gamma$. We start with the first one.

Assume that $x(b) \notin \Gamma$. In other words, $t = b$ is not the discontinuity point of $x^0(t)$. From $C7)$ it implies that $x(b) \notin \partial\Gamma$ too. For a positive number $\alpha \in R$ we shall construct a set G^α in the following way. Let
$F_\alpha = \{(t,x) | t \in [0,b], \|x - x^0(t)\| < \alpha\}, G_i(\alpha), i = \overline{0, m+1}$, be α-neighbourhoods of points $(0, x_0), (\theta_i, x(\theta_i)), i = \overline{1,m}, (b, x^0(b))$ in $\mathbb{R} \times \mathbb{R}^n$ respectively, and $\bar{G}_i(\alpha), i = \overline{1,m}$, be α-neighbourhoods of points $(\theta_i, x^0(\theta_i+))$ respectively. Denote

$$G^\alpha = F_\alpha \cup \left(\cup_{i=0}^{m+1} G_i(\alpha)\right) \cup \left(\cup_{i=1}^{m} \bar{G}_i(\alpha)\right)$$

Take $\alpha = h$ sufficiently small so that $G^h \subset G_t \times G_x$ where G_t is an interval such that $[0,b] \subset G_t$.
Fix $\epsilon \in \mathbb{R}$, $0 < \epsilon < h$.

1. In view of the theorem on continuous dependence on parameters [36] there exists $\bar{\delta}_m \in R, 0 < \bar{\delta}_m < \epsilon$, such that $\bar{G_m}\left(\bar{\delta}_m\right) \cap \Gamma = \emptyset$ and every solution $x_m(t)$ of (2), which starts in $\bar{G_m}(\bar{\delta}_m)$, is continuable to $t = b$, does not intersect Γ and
$$\left\|x_m(t) - x^0(t)\right\| < \epsilon$$
for those t.

2. The continuity of J implies that there exists $\delta_m \in R$, $0 < \delta_m < \epsilon$, such that $(\kappa, x) \in G_m(\alpha_m)$ implies $(\kappa, x + J(x)) \in \bar{G}_m(\bar{\alpha}_m) \cap D$.

3. Using corollary 6.1, continuous dependence of solutions on initial data, one can find $\bar{\delta}_{m-1}, 0 < \bar{\delta}_{m-1} < \epsilon$, such that a solution $x_{m-1}(t)$ of (2), which starts in $\bar{G}_{m-1}(\bar{\alpha}_{m-1}) \cap D$, $\bar{G}_{m-1}(\bar{\alpha}_{m-1}) \cap \Gamma \neq \emptyset$, intersects Γ in $G_m(\alpha_m)$ (we continue the solution $x_{m-1}(t)$ only to the moment of the intersection) and $\|x_{m-1}(t) - x^0(t)\| < \epsilon$ for all t from the common domain of $x_{m-1}(t)$ and $x^0(t)$.

Continuing the process for $m-2, m-3, \ldots, 1$, one can obtain a sequence of families of solutions of (2) $x_i(t), i = \overline{1, m}$, and a number $\delta \in \mathbb{R}, 0 < \delta < \epsilon$, such that a solution $x(t) = x(t, 0, \bar{x})$, which starts in $G_0(\delta) \cap D$, coincides over the first interval of continuity, except possibly, the δ_1-neighbourhood of θ_1, with one of the solutions $x_1(t)$. Then on the interval $[\theta_1, \theta_2]$ it coincides with one of the solutions $x_2(t)$, except possibly, the δ_1-neighbourhood of θ_1 and the δ_2-neighbourhood of θ_2, etc. Finally, one can see that the integral curve of $x(t)$ belongs to G^ϵ, it has exactly meeting k points with Γ, θ_i^1, $i = \overline{1, m}$, $|\theta_i^1 - \theta_i| < \epsilon$ for all i and is continuable to $t = b$.

If $x(b) \in \Gamma$, then it is easy to see that $x(t)$ has either a discontinuity points $\theta_m^1 < \theta_m$ or only $m-1$ discontinuity point $\theta_i^1, i = \overline{1, m-1}$ in $[0, b]$.

Assume that $x_0 \in \Gamma$. In this case $t = 0$ is a jump moment for $x^0(t)$ and $x(0+) \notin \Gamma$, that is $0 = \theta_1$. We assume that $x^0(t)$ has points of discontinuity $\theta_i, i = \overline{1,m}$. Similarly to the previous case one can find the $\bar{\delta}_1$-neigbourhood $\bar{G}(\bar{\delta}_1)$ of the point $(\theta_1, x(\theta_1^+))$ which serves the same role as $\bar{\delta}_1$ in the first case. That is, if $(\kappa, x) \in \bar{G}(\bar{\delta}_1) \cap D$ then the solution $x(t)$ belongs to the $\epsilon-$ neighbourhood of $x^0(t)$ in $B_{[a,b]}-$ topology. Now, using condition C5) and continuity of f and J, it is easy to find $\delta, 0 < \delta < \epsilon$, such that every solution $x(t)$ of (1) which starts in $\delta-$ neighbourhood of $(0, x_0)$ in D, intersects Γ in $G_1(\delta_1) \cap D$.

For the case $a \leq t \leq 0$ we only should remark that similarly to $0 \leq t \leq b$ for a given $\epsilon > 0$ one can find δ' such that $(0, \bar{x}) \in G_0(\delta')$ implies that $x(t, 0, \bar{x})$ is in the ϵ-neighbourhood of $x^0(t)$ in $B[a, 0]$-topology. Finally, if $\delta(\epsilon) = \min(\delta, \delta')$ and $(0, \bar{x}) \in G_0(\delta(\epsilon))$ then $x(t, 0, \bar{x})$ is in the $\epsilon-$neigbourhood of $x^0(t)$ in $B[a, b]-$topology. The theorem is proved.

6 $B-$equivalence

Let us introduce the following functions $\tau = \tau(x)$ and $\Psi = \Psi(x)$ which will be needed throughout the rest of the paper. Fix $\kappa \in \mathbb{R}$. Denote by $x(t) = x(t, \kappa, x)$ a solution of (2), $\tau = \tau(x)$ the moment of the first meeting of $x(t)$ with the surface Γ as t increases or decreases and $\tilde{\tau} = \tilde{\tau}(x)$ the moment of the first meeting of $x(t)$ with the surface $\tilde{\Gamma}$ as t increases or decreases.

LEMMA 6.1 $\tau(x), \tilde{\tau}(x) \in C^1$.

Proof. Let us consider

τ as for $\tilde{\tau}$ the proof is similar. Differentiating $\Phi(x(\tau, \kappa, x)) = 0$, and using $C5)$ one can get that

$$\frac{\partial \Phi(x(\tau, \kappa, x))}{\partial \tau} = \frac{\partial \Phi(x(\tau, \kappa, x))}{\partial x} \frac{dx(t)}{dt}\Big|_{t=\tau} = \frac{\partial \Phi(x(\tau, \kappa, x))}{\partial x} f(x(\tau, \kappa, x)) \neq 0$$

The proof of the lemma follows immediately from the implicit function theorem and conditions on (2).

Corollary 6.1 $\tau(x), \tilde{\tau}(x)$ *are continuous functions.*

Now let $x_1 = x(t, \tau, x(\tau)) + J(x(\tau)), \tilde{x}_1 = x(t, \tilde{\tau}, x(\tilde{\tau})) + J^{-1}(x(\tilde{\tau}))$ be also solutions of (2). Define functions $\Psi(x) = x_1(\kappa), \tilde{\Psi}(x) = \tilde{x}_1(\kappa)$.
Similarly to Lemma 6.1, one can show that the following assertion is valid.

Lemma 6.2 $\Psi(x), \tilde{\Psi}(x) \in C^1$

Consider the solution $x^0(t) : [a, b] \to R^n, a \leq 0 \leq b$, of (1) again. This time we assume that all points of discontinuity $\{\theta_i\}$ are interior points of $[a, b]$. That is, $a < \theta_{-k}$ and $\theta_m < b$.

The following system of impulsive differential equations is very important in sequel

$$
\begin{aligned}
& y(t) = f(y), \quad t \neq \theta_i, \\
& y(\theta_i+) = W_i(y(\theta_i), \text{for}\, i > 0, \\
& y(\theta_i) = \tilde{W}_i(y(\theta_i+), \text{for}\, i < 0,
\end{aligned}
\tag{7}
$$

where the function f is the same as in (1) and the maps $W_i, \tilde{W}_i$ will be defined below. The following condition will be nedeed in the rest of our paper. Without loss of generality, assume that there exists $r_1 \in \mathbb{R}, 0 < r_1 < r$, such that the r_1- neighbourhoods $G_i(r_1)$ of $(\theta_i, x^0(\theta_i))$ do not intersect each other. In view of $C5$), one can suppose that r_1 is sufficiently small so that every solution of (2) which starts in $G_i(r_1)$ intersects Γ in $G_i(r_1)$ exactly once as t increases or decreases.
Fix $i = 1, \ldots, m$ and let $\xi(t) = x(t, \theta_i, x), (\theta_i, x) \in G_i(r_1)$, be a solution of (2) and $\tau_i = \tau_i(x)$,$\tau_i \geq \theta_i$ or $\tau_i < \theta_i$, be a meeting time of $\xi(t)$ with Γ and $\psi(t) = x(t, \tau_i, \xi(\tau_i) + J(\xi(\tau_i))$ be another solution of (2). Denote $W_i(x) = \psi(\theta_i)$.

One can see that

$$W_i(x) = x + \int_{\theta_i}^{\tau_i} f(\xi(s))ds + J(x + \int_{\theta_i}^{\tau_i} f(\xi(s))ds) + \int_{\tau_i}^{\theta_i} f(\psi(s))ds \tag{8}$$

is a map of an intersection of the plane $t = \theta_i$ with $G_i(r_1)$ into the plane $t = \theta_i$. Similarly for $i = -k, \ldots, -1$, if we denote by $\xi(t) = x(t, \theta_i, x)$ and $\psi(t) = x(t, \tilde{\tau}_i, \xi(\tilde{\tau}_i) + J^{-1}(\xi(\tilde{\tau}_i))$ corresponding solutions of (2), then

$$\tilde{W}_i(x) = x + \int_{\theta_i}^{\tilde{\tau}_i} f(\xi(s))ds + J^{-1}(x + \int_{\theta_i}^{\tilde{\tau}_i} f(\xi(s))ds) + \int_{\tilde{\tau}_i}^{\theta_i} f(\psi(s))ds \tag{9}$$

The functions $W_i, \tilde{W}_i$ are the maps Ψ and $\tilde{\Psi}$ respectively defined in the beginning of this section with $\kappa = \theta_i$. Hence, Lemma 6.2 implies that all $W_i, \tilde{W}_i$ are continuously differentiable maps. It is obvious, that for sufficiently small r_1, $W_i(x), \tilde{W}_i(x) \in G_r$. Further, $(\alpha \hat{,} \beta], \{\alpha, \beta\} \subset R$, stands for an oriented interval, that is

$$(\alpha \hat{,} \beta] = \begin{cases} (\alpha, \beta] & \text{if } \alpha \leq \beta \\ (\beta, \alpha], & \text{otherwise.} \end{cases}$$

Let $x(t)$ be a solution of (1), $x(t) = x(t, a, x(a))$, and $x(t)$ be close to $x^0(t)$ in $B_{[a,b]}-$ topology so that x(t) has exactly $m-k$ points $\tau_i, i = -k, \ldots, -1, 1, 2, \ldots, m$, of discontinuity in $[a, b]$. Denote by $G(h)$ an $h-$neighbourhood of the point $x^0(0)$.

Definition 6.1 *The systems (1) and (7) are said to be $B-$ equivalent in G^{r_1} if there exists $h \in R, 0 < h$, such that:*

1. *for every solution $x(t)$, such that $x(0) \in G(h)$, the integral curve of $x(t)$ belongs to G^{r_1} and there exists a solution $y(t) = y(t, 0, x(0))$ of (7) which satisfies*

$$x(t) = y(t), t \in [a, b] \setminus \cap_{i=-k}^{m} (\tau_i \hat{,} \theta_i]. \tag{10}$$

Particularly,

$$\begin{aligned} x(\theta_i) &= \begin{cases} y(\theta_i), & \text{if } \theta_i \leq \tau_i, \\ y(\theta_i^+), & \text{otherwise,} \end{cases} \\ y(\tau_i) &= \begin{cases} x(\tau_i), & \text{if } \theta_i \geq \tau_i, \\ x(\tau_i^+), & \text{otherwise.} \end{cases} \end{aligned} \tag{11}$$

2. *Conversely, if (7) has a solution* $y(t) = y(t, 0, x(0)), x(0) \in G(h)$, *then there exists a solution* $x(t) = x(t, 0, x(0))$ *of (1) which has an integral curve in* G^{r_1}, *and (11) holds.*

LEMMA 6.3 $x_0(t)$ *is a solution of (1) and (7) simultaneously.*

Proof. The proof follows immediately from (8) and (9).

THEOREM 6.1 *Assume that conditions* $C1) - C7)$ *are fulfilled. Then systems (1) and (7) are B-equivalent in* G^{r_1} *if* r_1 *is sufficiently small.*

Proof. Assume that $r_1 > 0$ is sufficiently small so that $W_i, i = -k, \ldots, -1, 1, \ldots, m$, are defined. Let us check only the first part of Definition 6.1 as for the second one the proof is analogous. Theorem 5.1 implies that there exists a small $h, 0 < h < r_1$, such that if $\|\bar{x} - x_0\| < h$ and $\bar{x} \in D$, then the solution $x(t) = x(t, 0, \bar{x})$ belongs to $G^{r_1} \cap G_t \times D$, where $r_1 > 0$ has been chosen for W_i above. Assume that h is sufficiently small so that $x(t)$ has exactly $m + k - 1$ moments of discontinuity $t = \tau_i, i = -k, \ldots, -1, 1, \ldots, m$. Without loss of generality, we suppose that $\theta_i > \tau_i$ for all i and $x(0)$ is not the point of discontinuity. It is obvious that we need only to prove the theorem for $[0, b]$, as for $[a, 0]$ the proof is similar. Consider the solution $y(t) = x(t, 0, x(0))$ of (7). By the theorem on existence and uniqueness [36] the equality

$$x(t) = y(t) \tag{12}$$

on $[0, \tau_1]$ is valid. Since $(\tau_1, x(\tau_1)) \in G^{r_1}$ we have that

$$y(\theta_1+) = \int_{\tau_1}^{\theta_1} f(y(s))ds + W_i(y(\theta_1)). \tag{13}$$

is defined and, moreover,

$$x(\theta_1) = x(\tau_1) + J(x(\tau_1)) + \int_{\tau_1}^{\theta_1} f(x(s))ds. \tag{14}$$

Using (12)-(14) one can obtain that

$$y(\theta_1+) = x(\tau_1) + \int_{\tau_1}^{\theta_1} f(y(s))ds + \int_{\theta_1}^{\tau_1} f(y(s))ds$$
$$+J(y(\tau_1)) + \int_{\tau_1}^{\theta_1} f(x(s))ds = x(\theta_1).$$

Now, defining $x(t)$ and $y(t)$ as solutions of (2) with a common initial value $x(\theta_1)$, one can see that $x(t) = y(t), t \in (\theta_1, \tau_2]$. Continuing in the same manner for all $t \in [0, b]$ one can show that $y(t)$ is continuable to $t = b$ and (10) holds. Moreover, it is easily seen that for sufficiently small r_1 the integral curve of $y(t)$ belongs to G_r. The theorem is proved.

7 Differentiability of solutions in initial value

Let us define derivatives of functions $\tau_i(x), W_i(x), i = 1\ldots, m$, and $\tilde{\tau}_i(x), \tilde{W}_i(x), i = -k, \ldots, -1$, which were described in Section 6, at the points $(x^0(\theta_i))$ and $(x^0(\theta_i+))$ respectively. We start with derivatives of $\tau_i(x)$ and $\tilde{\tau}_i(x)$. One should emphasize that $\tau_i, \tilde{\tau}_i$ are maps $\tau, \tilde{\tau}$ defined in Section 6 with $\kappa = \theta_i$. The equalities $\Phi(x(\tau_i(x))) = 0, \tilde{\Phi}(x(\tilde{\tau}_i(x))) = 0$ imply that

$$\Phi_x(x^0(\theta_i))f(x^0(\theta_i))d\tau_i + \sum_{j=1}^{n} \Phi_x(x^0(\theta_i))\frac{\partial x^0(\theta_i)}{\partial x_j}dx_j,$$
$$\tilde{\Phi}_x(x^0(\theta_i+))f(x^0(\theta_i+))d\tau_i + \sum_{j=1}^{n} \tilde{\Phi}_x(x^0(\theta_i+))\frac{\partial x^0(\theta_i+)}{\partial x_j}dx_j.$$

Using the last expression, one can obtain that

$$\frac{\partial \tau_i(x^0(\theta_i))}{\partial x_j} = -\frac{\Phi_x(x^0(\theta_i))\frac{\partial x^0(\theta_i)}{\partial x_j}}{\Phi_x(x^0(\theta_i))f(x^0(\theta_i))},$$
$$\frac{\partial \tilde{\tau}_i(x^0(\theta_i+))}{\partial x_j} = -\frac{\partial \Phi_x(x^0(\theta_i+))\frac{\partial x^0(\theta_i+)}{\partial x_j}}{\tilde{\Phi}_x(x^0(\theta_i+))f(x^0(\theta_i+))}. \tag{15}$$

Similarly, for W_i the following expression is valid

$$\frac{\partial W_i(x^0(\theta_i))}{\partial x_j} = e_j + f\frac{\partial \tau_i}{\partial x_j} + \frac{\partial J}{\partial x}(e_j + f\frac{\partial \tau_i}{\partial x_j}) - f^+\frac{\partial \tau_i}{\partial x_j},$$
$$\frac{\partial \tilde{W}_i(x^0(\theta_i+))}{\partial x_j} = e_j + f^+\frac{\partial \tilde{\tau}_i}{\partial x_j} + \frac{\partial J^{-1}}{\partial x}(e_j + f\frac{\partial \tilde{\tau}_i}{\partial x_j}) - f^+\frac{\partial \tilde{\tau}_i}{\partial x_j}, \tag{16}$$

where $e_j = (0, \ldots, 1, \ldots, 0)$ and the unit is $j-$th coordinate. Assume that $x^0(t) : [a, b] \to \mathbb{R}^n$ is the solution of (1) and (7). Moreover, systems (1) and (7) are $B-$equivalent in G^r and there exists $\delta \in R, \delta > 0$, such that every solution which starts in $G_0(\delta)$ is continuable to $t = b$. Without loss of generality, assume that all points of discontinuity of $x^0(t)$ are interior. Denote by $x_j(t), j = \overline{1,n}$, a solution of (1) such that $x_j(t_0) = x_0 + \xi e_j = (x_1^0, x_2^0, \ldots, x_{j-1}^0, x_j^0 + \xi, x_{j+1}^0, \ldots, x_n^0), \xi \in \mathbb{R}$, $(t_0, x_0 + \xi e_j, \mu_0) \in C_0(\delta)$ and let θ_i^j be the moments of discontinuity of $x_j(t)$. By Theorem 5.1, for sufficiently small $|\xi|$ the solution $x_j(t)$ is defined on $[a, b]$.

Definition 7.1 *The solution $x^0(t)$ is said to be differentiable in $x_j^0, j = \overline{1,n}$, if*

A) there exist such constants $\nu_{ij}, i = -k, \ldots, -1, 1, \ldots, m$, that

$$\theta_i^j - \theta_i = \nu_{ij}\xi + o(|\xi|); \tag{17}$$

B) for all $t \in [a, b] \setminus \cup_{i=-k}^m \widehat{(\theta_i, \theta_i^j]}$, the following equality is satisfied

$$x_j(t) - x^0(t) = u_j(t)\xi + o(|\xi|), \tag{18}$$

where $u_j(t)$ is a piecewise continuous function, with discontinuities of the first kind at the points $t = \theta_i, i = -k, \ldots, -1, 1, \ldots, m$.

The pair $\{u_j, \{\nu_{ij}\}_i\}$ is said to be a $B-$ derivative of $x^0(t)$ in initial value x_0^j on $[a, b]$.

Lemma 7.1 *Assume that conditions $C1) - C7)$, then the solution $x^0(t)$ of (7) has $B-$ derivatives in the initial value on $[a, b]$. Moreover:*

1) $u_j, j = \overline{1,n}$, are solutions of the linear system

$$\begin{aligned}
&\frac{du}{dt} = f_x(x^0(t))u, \quad t \neq \theta_i, \\
&u(\theta_i+) = W_{ix}(x^0(\theta_i))u(\theta_i), if i > 0, \\
&u(\theta_i) = \tilde{W}_{ix}(x^0(\theta_i+))u(\theta_i+), if i < 0,
\end{aligned} \tag{19}$$

with the initial conditions $u(t_0) = e_j, j = \overline{1,n}$, respectively and constants $\nu_{ij} = 0$, for all i, j.

Proof. Fix $p = \overline{1,n}$. We shall prove the Lemma only for the derivative in x_0^p and for $t \geq 0$. Let $y_p(t) = y(t, t_0, x_0+\xi e_p, \mu_0)$. By the theorem on differentiability with respect to parameters [36] we have that $y_p(t) - x^0(t) = u_p(t)\xi + \rho(\xi), \quad \rho(\xi) = o(|\xi|)$, for all $t \in [0, \theta_1]$. Particularly, $y_p(\theta_1) - x^0(\theta_1) = u_p(\theta_1)\xi + \rho(|\xi|)$. Then $y_p(\theta_1+) - x^0(\theta_1+) = W_1(y_p(\theta_1)) - W_1(x^0(\theta_1)) = W_{1x}(x^0(\theta_1)))(u_p(\theta_1)\xi + \rho(\xi)] + \bar{\rho}_1(\xi)$. Since $\bar{\rho}_1 = o(|\xi|)$, we have that $y_p(\theta_1+) - x^0(\theta_1+) = u_p(\theta_1+)\xi + \tilde{\rho}_1(\xi)$, where $\tilde{\rho}_1 = o(|\xi|)$. Denote by $U(t), U(\theta_1) = I$, the fundamental matrix of solutions of the system $u'(t) = f_x(x^0(t))$. Using the theorem from [36] again one can obtain that for all $t \in (\theta_1, \theta_2]$ the following relation is true $y_p(t) - x^0(t) = U(t)(y_p(\theta_1+) - x^0(\theta_1+)) + \rho(y_p(\theta_1+) - x^0(\theta_1+)) = U(t)u_p(\theta_m+)\xi + \rho_2(\xi) = u_p(t)\xi + \rho_2(\xi)$, where $\rho_2 = o(|\xi|)$. Continuing the process we can prove that (18) is valid. Formula (17) involving constants ν_i^j is trivial. The Lemma is proved.

THEOREM 7.1 *Assume that conditions* $C1) - C7)$ *are satisfied. Then the solution* $x^0(t)$ *of (1) has* $B-$ *derivatives in the initial value on* $[a, b]$. *Moreover:*

$u_j(t), j = \overline{1,n}$, *are respectively solutions of the equation (19) with the initial conditions* $u(t_0) = e_j, j = \overline{1,n}$, *and*

$$\nu_{ij} = -\frac{\Phi_x u_j(\theta_i)}{\Phi_x f}, j = \overline{1,n}, i = \overline{1,m}, \nu_{ij} = -\frac{\tilde{\Phi}_x(x^0(\theta_i+))u_j(\theta_i+)}{\tilde{\Phi}_x(x^0(\theta_i+))f(x^0(\theta_i+))}, j = \overline{1,n}, i = \overline{-k,-1}.$$

The proof of the theorem follows immediately from Theorem 6.1, Lemma 7.1 and formulas (15), (16).

Remark 7.1 Higher order smoothness of DDS is considered in [31].

8 Conclusion

Let $G \subset \mathbb{R}^n$ be an open set and $\Gamma, \tilde{\Gamma}$ be disjoint subsets of $\bar{G}$. Denote $D = G \cup \Gamma \cup \tilde{\Gamma}$.

DEFINITION 8.1 *We say that a* $B-$ *smooth DF is a map* $\phi : \mathbb{R} \times D \to D$, *which satisfies the following properties:*

I) *The group property:*

(i) $\phi(0,x): D \to D$ *is the identity;*

(ii) $\phi(t,\phi(s,x)) = \phi(t+s,x),$ *is valid for all* $t,s \in \mathbb{R}$ *and* $x \in D.$

II) *If* $x \in D$ *is fixed then* $\phi(t,x) \in \mathcal{PC}^1(\mathbb{R}),$ *and* $\phi(\theta_i,x) \in \Gamma, \phi(\theta_i+,x) \in \tilde{\Gamma}$ *for every discontinuity point* θ_i *of* $\phi(t,x).$

III) *The function* $\phi(t,x)$ *is* $B-$ *differentiable in* $x \in D$ *on* $[a,b] \subset \mathbb{R}$ *for every* $\{a,b\} \subset \mathbb{R},$ *assuming that all discontinuity points of* $\phi(t,x)$ *are interior points of* $[a,b].$

One can see that the system (1) defines a $B-$ smooth DF provided conditions $C1) - C8)$ and the conditions of one of the continuation theorems are fulfilled.

DEFINITION 8.2 *We say that a* DF *is a map* $\phi : \mathbb{R} \times D \to D,$ *which satisfies the properties* $I), II)$ *of Definition 8.1 and the following conditions are valid:*

IV) *If* $x \in D$ *is fixed then* $\phi(t,x) \in \mathcal{PC}(\mathbb{R}),$ *and* $\phi(\theta_i,x) \in \Gamma, \phi(\theta_i+,x) \in \tilde{\Gamma}$ *for every discontinuity point* θ_i *of* $\phi(t,x).$

V) *The function* $\phi(t,x)$ *is* $B-$ *continuous in* $x \in D$ *on* $[a,b] \subset \mathbb{R}$ *for every* $\{a,b\} \subset \mathbb{R}.$

Comparing definitions of the $B-$ differentiability and the $B-$ continuity one can conclude that every $B-$ smooth DF is a $DF.$

9 Examples

EXAMPLE 9.1 *Consider the following impulsive differential equation*

$$x_1' = \alpha x_1 - \beta x_2,\; x_2' = \beta x_1 + \alpha x_2, \textit{if } (x(t) \notin \Gamma \wedge t \geq 0) \vee (x(t) \notin \tilde{\Gamma} \wedge t \leq 0),$$
$$x_1(t+) = \sqrt{3}x_1(t-) - x_2(t-),\; x_2(t+) = x_1(t-) + \sqrt{3}x_2(t-), \textit{if } x(t) \in \Gamma \wedge t \geq 0,$$
$$x_1(t-) = \frac{\sqrt{3}}{4}x_1(t+) + \frac{1}{4}x_2(t+),\; x_2(t-) = -\frac{1}{4}x_1(t+) + \frac{\sqrt{3}}{4}x_2(t+), \textit{if } x(t) \in \tilde{\Gamma} \wedge t \leq 0,$$

where $\Gamma = \{(x_1,x_2)|\, x_2 = \frac{1}{2}x_1, x_1 > 0\}, \tilde{\Gamma} = \{(x_1,x_2)|\, x_2 = \frac{\sqrt{3}}{2}x_1, x_1 > 0\}$, *constants* α, β *are positive. One can see that* $\Phi(x) = x_2 - \frac{1}{2}x_1, f(x) = (\alpha x_1 - \beta x_2, \beta x_1 + \alpha x_2), J(x) = (\sqrt{3}x_1 - x_2, x_1 + \sqrt{3}x_2)$. *We assume that* $G = \{(x_1,x_2)|\frac{1}{2}x_1 < x_2 < \frac{\sqrt{3}}{2}x_1, x_1 > 0\}$. *One can verify that the functions and the sets satisfy* $C1) - C7)$. *Let us check if the conditions of Theorem 3.3 hold. Fix* $x \in \tilde{\Gamma}$. *Then* $dist(x,\Gamma) = \frac{1}{2}||x||$ *and* $||f(x)|| = \sqrt{(\alpha x_1 - \beta x_2)^2 + (\beta x_1 + \alpha x_2)^2} = \sqrt{\alpha^2+\beta^2}||x||$. *Thus* $\sup ||f||_{B(x,\epsilon_x)} = \sqrt{\alpha^2+\beta^2}(||x|| + \frac{1}{2}||x||) = \frac{3}{2}\sqrt{\alpha^2+\beta^2}||x||$, *and* $\inf_{\tilde{\Gamma}\times(0,\infty)} \frac{\epsilon_x}{\sup_{B(x,\epsilon_x)}||f||} = \frac{2}{3\sqrt{\alpha^2+\beta^2}} > 0$. *Hence, all conditions of a DF for the system are fulfilled.*

Example 9.2 *Consider the following model for simple neural nets from [3]. We have modified its form according to the proposed equation (1).*

$$x_1' = x_2,\, x_2' = -\beta^2 x_1,\, p' = -\gamma p + x_1 + B_0, if\ (x(t) \notin \Gamma \wedge t \geq 0) \vee (x(t) \notin \tilde{\Gamma} \wedge t \leq 0),$$

$$x_1(t+) = x_2(t-),\, x_2(t+) = x_2(t-),\, p(t+) = 0, if\ x(t) \in \Gamma \wedge t \geq 0,$$

$$x_1(t-) = x_1(t+),\, x_2(t-) = x_2(t+),\, p(t-) = r, if\ x(t) \in \tilde{\Gamma} \wedge t \leq 0,$$

where $\beta, B_0 \in \mathbb{R}$ *are constants,* $\Gamma = \{(x_1,x_2,p)|\, p = r\}, \tilde{\Gamma} = \{(x_1,x_2,p)|\, p = 0\}, \Phi(x) = p - r,\ f(x) = (x_2, \beta^2 x_1, -\gamma p + x_1 + B_0), J(x) = (x_1, x_2, r), \beta, \gamma, r > 0$, *are constants. We assume that* $G = \{(x_1,x_2,p)|0 < p < r, x_1^2 + \frac{x_2^2}{\beta^4} < 1\}$. *In the system the variable* $p(t)$ *is a scalar input of a neural trigger and* x_1, x_2, *are other variables. The value of* r *is the threshold. One can verify that the functions and the sets satisfy* $C1) - C7)$ *and the conditions of Theorem 3.4. That is, the system defines a DF.*

Example 9.3 *Let us consider the following discontinuous system*

$$x_1' = \alpha x_1 - \beta x_2,\, x_2' = \beta x_1 + \alpha x_2, if\ (x(t) \notin \Gamma \wedge t \geq 0) \vee (x(t) \notin \tilde{\Gamma} \wedge t \leq 0),$$

$$x_1(t+) = kx_1(t-),\, x_2(t+) = kx_2(t-), if\ x(t) \in \Gamma \wedge t \geq 0,$$

$$x_1(t-) = \frac{1}{k}x_1(t+),\, x_2(t-) = -\frac{1}{k}x_2(t+), if\ x(t) \in \tilde{\Gamma} \wedge t \leq 0, \tag{20}$$

where $\Gamma = \{(x_1, x_2)|\, x_1^2 + x_2^2 = r_1\}, \tilde{\Gamma} = \{(x_1, x_2)|\, x_1^2 + x_2^2 = kr_1\}, \alpha, \beta, k$ *are constants such that* $\alpha\beta < 0, 1 < k$. *Assume that* $G = \mathbb{R}^2$.

One can see that all conditions $C1) - C6)$ *are valid for the system, and so are conditions of Theorem 3.4. But* $C7)$ *is not fulfilled, and it is easy to see that a solution* $x(t, 0, x_0)$ *of (20), which starts outside of* $\tilde{\Gamma}$, *does not satisfy the condition* $x(-t, 0, x(t, 0, x_0)) = x_0$ *for all* t. *Thus (20) does not define a DF.*

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